MATHEMATICAL SYSTEMS

More than 5000 years ago, the Egyptians and Babylonians were using geometry in surveying and architecture. These ancient mathematicians discovered geometric facts and relationships through experimentation and inductive reasoning. Because of their approach, they could never be sure of their conclusions, and in some cases their formulas were incorrect. The ancient Greeks, on the other hand, viewed points, lines, and figures as abstract concepts about which they could reason deductively. They were willing to experiment in order to formulate ideas, but final acceptance of a mathematical statement depended on proof by deductive reasoning. The Greeks' approach was the beginning of mathematical systems.

A **mathematical system** consists of *undefined terms, definitions, axioms,* and *theorems.* There must always be some words that are undefined. *Line* is an example of an undefined term in geometry. We all have an intuitive idea of what a line is, but trying to define it involves more words, such as *straight, extends indefinitely,* and *has no thickness.* These words would also have to be defined. To avoid this problem of *circularity,* certain basic words such as *point* and *line* are **undefined terms.** These words are then used in **definitions** to define other words. Similarly, there must always be some statements, called **axioms,** that we assume to be true and do not try to prove. Finally, the axioms, definitions, and undefined terms are used together with deductive reasoning to prove statements called **theorems.**

Theorems

↑

Undefined terms
Definitions
Axioms

HISTORICAL HIGHLIGHT

Euclid, ca. 350 B.C.E.

The crowning achievement of Greek mathematical reasoning was Euclid's *Elements,* a series of 13 books written about 300 B.C.E. These books contain over 600 theorems, which were obtained by deductive reasoning from 10 basic assumptions called axioms. Although much of the material was drawn from earlier sources, the superbly logical arrangement of the theorems displays the genius of the author. Euclid's *Elements* stood as a model of deductive reasoning for over 2000 years, and few books have been more important to the thought and education of the western world.*

*D. M. Burton, *The History of Mathematics,* 6th ed. (New York: McGraw-Hill, 2006), pp. 143–170.

POINTS, LINES, AND PLANES

One fundamental notion in geometry is that of a *point.* All geometric figures are sets of points. **Points** are abstract ideas, which we illustrate by dots, corners of boxes, and tips of pointed objects. These concrete illustrations have width and thickness, but points have no dimensions. The following description of a point, from *Mr. Fortune's Maggot,* by Sylvia Townsend Warner, indicates some of the problems associated with teaching elementary school children the concept of a point.†

†Quoted in J. R. Newman, *The World of Mathematics,* 4th ed. (New York: Simon and Schuster, 1956), p. 2254.

Calm, methodical, with a mind prepared for the onset, he guided Lueli down to the beach and with a stick prodded a small hole in it.

"What is this?"

"A hole."

"No, Leuli, it may seem like a hole, but it is a point."

Perhaps he had prodded a little too emphatically. Lueli's mistake was quite natural. Anyhow, there were bound to be a few misunderstandings at the start. He took out his pocket knife and whittled the end of the stick. Then he tried again.

"What is this?"

"A smaller hole."

"Point," said Mr. Fortune suggestively.

"Yes, I mean a smaller point."

"No, not quite. It is a point, but it is not smaller. Holes may be of different sizes, but no point is larger or smaller than another point."

A **line** is a set of points that we describe intuitively as being "straight" and extending indefinitely in both directions. The edges of boxes and taut pieces of string or wire are models of lines. The line in Figure 9.1 passes through points A and B and is denoted by $\overleftrightarrow{AB}$. The arrows indicate that the line continues indefinitely in both directions. If two or more points are on the same line, they are called **collinear.**

Figure 9.1

A plane is another set of points that is undefined. We describe a **plane** as being "flat" like the top of a table, but extending indefinitely. The surfaces of floors and walls are other common models for portions of planes. A plane can be illustrated by a drawing that uses arrows, as in Figure 9.2, to indicate that it extends and is not bounded.

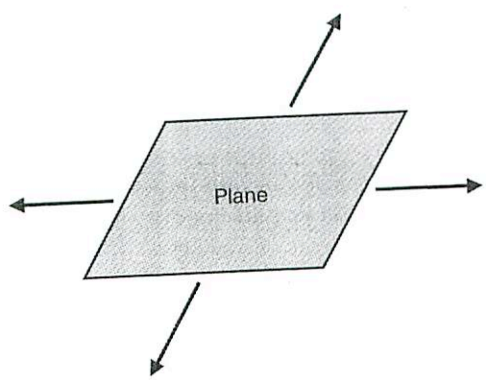


Figure 9.2

EXAMPLE A A standard sheet of paper is a model for part of a plane.

1. What part of a sheet of paper might be used as a model for a line?
2. What part of a sheet of paper might be used as a model for a point?
3. How can models of lines and points be obtained by folding a sheet of paper?

Solution **1.** Each edge of the paper is a model for part of a line. **2.** Each corner of the paper is a model for a point. **3.** The crease made by folding a sheet of paper is a model for part of a line. Two folds can produce parts of two lines that intersect in a point.

Points, lines, and *planes* are undefined terms in geometry that are used to define other terms and geometric figures. The following paragraphs contain some of the more common definitions and examples of figures that occur in planes.

HALF-PLANES, SEGMENTS, RAYS, AND ANGLES

Half-Planes A line in a plane partitions the plane into three disjoint sets: the points on the line and **two half-planes.** Line ℓ in Figure 9.3 partitions the plane into half-planes with point A in one half and point B in the other.

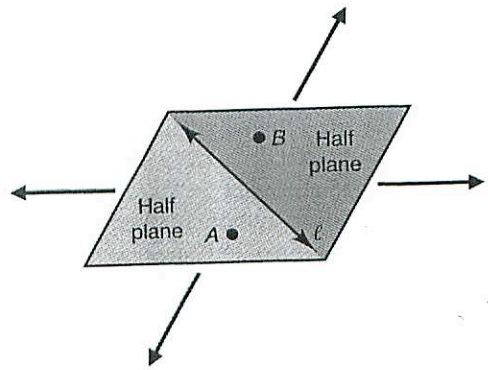


Figure 9.3

Line Segments A **line segment** consists of two points on a line and all the points between them (Figure 9.4). The line segment with **endpoints** A and B is denoted by $\overline{AB}$. To **bisect** a line segment means to divide it into two parts of equal length. The **midpoint** C bisects $\overline{AB}$.

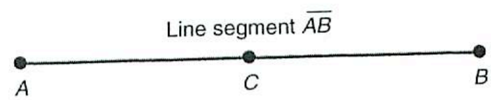


Figure 9.4

Half-Lines and Rays A point on a line partitions the line into three disjoint sets: the point and two **half-lines.** Figure 9.5a shows two half-lines that are determined by point P. A **ray** consists of a point on a line and all the points in one of the half-lines determined by the point. The ray in part b, which has D as an *endpoint* and contains point E, is denoted by $\overrightarrow{DE}$.

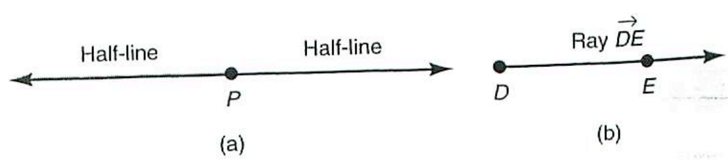


Figure 9.5

Angles An **angle** is formed by the union of two rays, as shown in Figure 9.6a, or by two line segments that have a common endpoint, as in part b. This endpoint is called the **vertex,** and the rays or line segments are called the **sides of the angle.** The angle with vertex G, whose sides contain points F and H, is denoted by $\measuredangle FGH$. Sometimes it is convenient to identify an angle by the letter of its vertex, such as $\measuredangle G$ in part a, or by a numeral, such as $\measuredangle 1$ in part b.

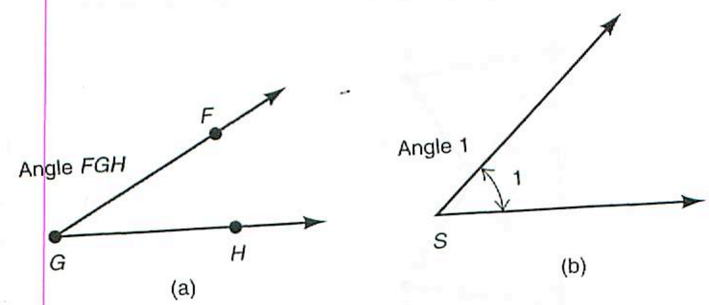


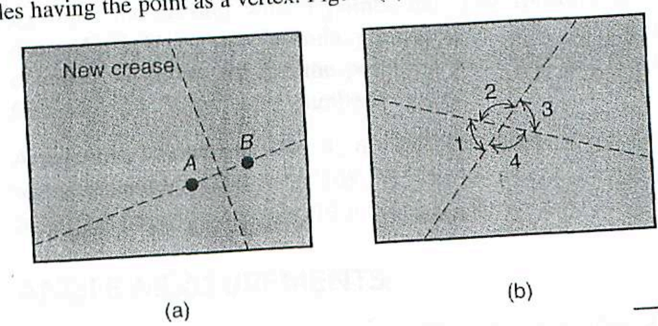
Figure 9.6

EXAMPLE B

Fold a standard sheet of paper to create models of the following terms.

1. Parts of two opposite half-planes
2. A bisected line segment
3. Part of a ray
4. An angle

Solution **1.** Any crease creates two half-planes. **2.** Fold the paper to obtain a crease and draw a line in the crease, as shown in figure (a). Select two points A and B on the line, and fold the line onto itself so that point A coincides with point B. The point where the new crease intersects segment $\overline{AB}$ is the midpoint that bisects $\overline{AB}$ into two segments. **3.** Any crease creates a line, and selecting a point on the line determines two rays. **4.** Any two folds that form creases that intersect in a point create four angles having the point as a vertex. Figure (b) shows angles 1, 2, 3, and 4.



PROBLEM-SOLVING APPLICATION

The ability to determine the number of line segments whose endpoints are a given number of points has many practical applications. One of these became evident in the early days of the development of the telephone system. The fundamental problem was how to connect two people who wanted to talk. This was done by connecting cords and plugs for each pair of people. In 1884, Ezra T. Gilliland devised a mechanical system that would allow 15 subscribers to reach one another without the aid of an operator.

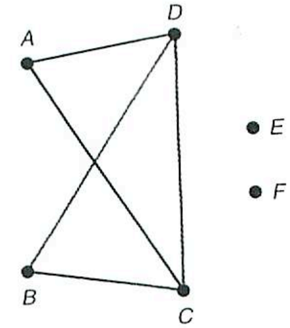
Problem

How many line segments are needed to connect 15 points in a plane so that each pair of points are the endpoints of a line segment?

Understanding the Problem One line segment connects 2 points, and 3 line segments connect 3 points. **Question 1:** How many line segments are needed to connect 4 points?

Devising a Plan Let's examine a few more special cases. Perhaps the strategies of *solving a simpler problem* and *finding a pattern* will lead to a solution. In the following figure, 6 line segments have the points A, B, C, and D as endpoints. **Question 2:** How many new

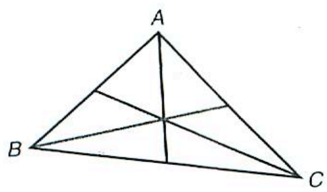
line segments are needed to connect E to each of these 4 points, and what is the total number of line segments connecting the 5 points?



Technology Connection

Properties of Triangles

If each vertex of a triangle is connected to the midpoint of the opposite side of the triangle, will the areas of the six smaller triangles ever be equal? This and similar questions are explored using Geometer's Sketchpad® student modules available at the companion website.



Mathematics Investigation
Chapter 9, Section 1
www.mhhe.com/bbn

Carrying Out the Plan Placing a sixth point, F, in the diagram, we can see that there will be 5 new line segments from F to the other points and a total of 15 line segments for the 6 points. Find a pattern and complete the following table. **Question 3:** How many line segments are required to connect 15 points?

No. of points	2	3	4	5	6	7	8	9	10	15
No. of segments	1	3	6	10	15					

Looking Back You probably recognize the numbers 1, 3, 6, 10, 15, etc., in the table as triangular numbers (Chapter 1). Note that the first triangular number is associated with 2 points, the second with 3 points, etc. The formula for the nth triangular number is $n(n + 1)/2$. Using this formula, you can determine the number of line segments needed to connect 20 points so that the points in each pair are the endpoints of a line segment. **Question 4:** What is the number?

Answers to Questions 1–4 **1.** 6 **2.** There will be 4 new line segments and a total of 10 line segments for the 5 points. **3.** 105 **4.** The number of line segments needed to connect 20 points is the 19th triangular number: $(19 \times 20)/2 = 190$.

ANGLE MEASUREMENTS

The ancient Babylonians devised a method for measuring angles by dividing a circle into 360 equal parts, called **degrees.** One degree (1°) is $\frac{1}{360}$ of a complete turn about a circle, as shown in Figure 9.7. Each degree can be divided into 60 equal parts, called **minutes,** and

NCTM Standards

Students in grades 3–5 can begin to establish some benchmarks by which to estimate or judge the size of objects. For example, they can learn that a "square corner" is called a *right angle* and establish this as a benchmark for estimating the size of other angles. p. 172

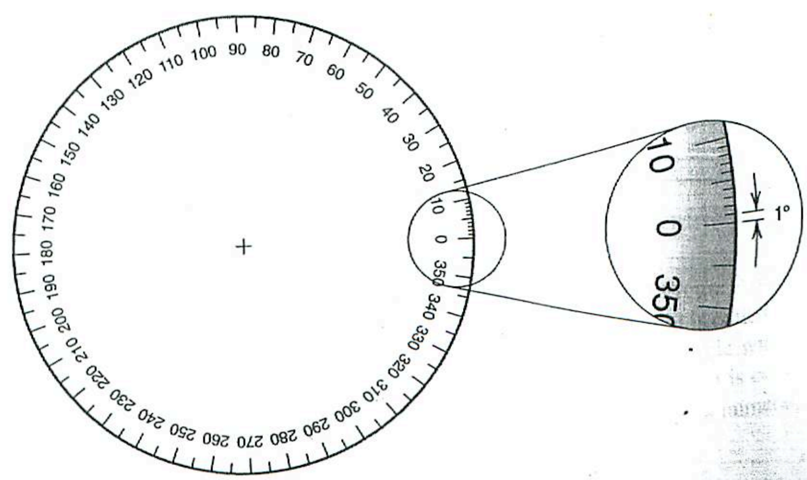


Figure 9.7

each minute can be divided into 60 equal parts, called **seconds.** This is the origin of the modern practice of dividing hours into minutes and seconds.

A **protractor** is a device for measuring angles (Figure 9.8). To **measure** an angle, place the center of the protractor on the vertex of the angle (B in this example), and line up one side of the angle ($\overline{BC}$) with the baseline of the protractor. The protractor in Figure 9.8 shows that $\measuredangle ABC$ has a measure of approximately 60°.

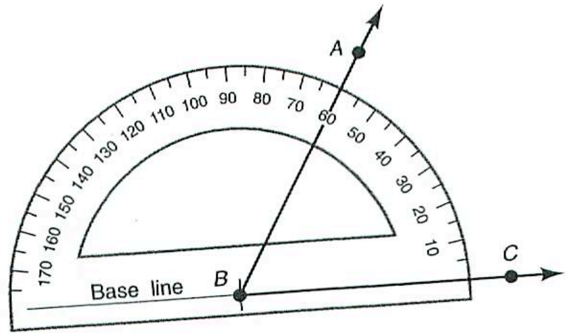


Figure 9.8

If an angle has a measure of 90°, as in Figure 9.9a, it is called a **right angle;** if it is less than 90°, and greater than 0°, as in part b, it is called an **acute angle;** if it is greater than 90° and less than 180°, as in part c, it is called an **obtuse angle;** and if it has a measure of 180° it is called a **straight angle.** It is customary to draw ⌉ at the vertex of a right angle. Occasionally we use angles with measures of more than 180° and less than 360°, as shown in Figure 9.9d. Such an angle is called a **reflex angle.** To indicate a reflex angle, we draw a circular arc to connect the two sides of the angle.

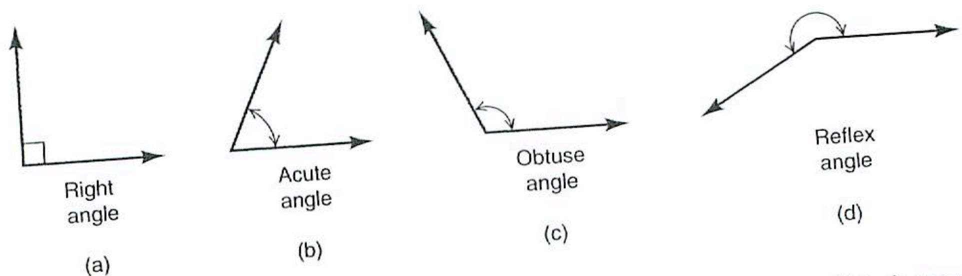


Figure 9.9

If the sum of two angles is 90°, the angles are called **complementary;** if their sum is 180°, they are called **supplementary.** Figure 9.10 shows special cases of complementary and supplementary angles in which the pairs of angles share a common side. If two angles have the same vertex, share a common side, and do not overlap, they are called **adjacent angles.** Angles 1 and 2 are adjacent complementary angles, and angles 3 and 4 are adjacent supplementary angles.

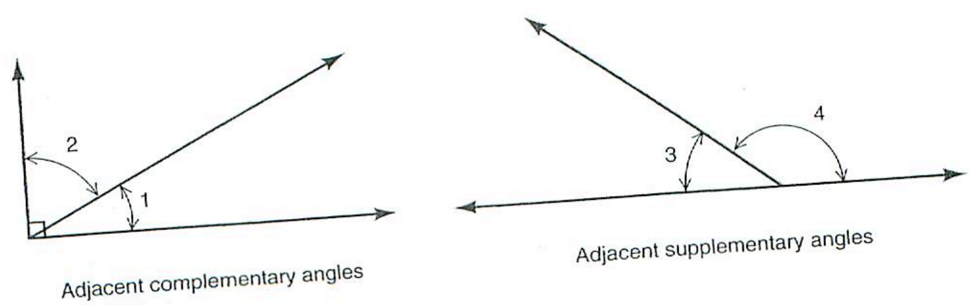


Figure 9.10

EXAMPLE C

Fold a standard sheet of paper to create models for the following terms.

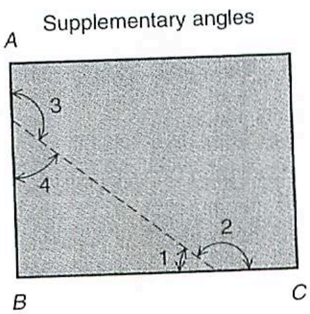
1. Acute angle
2. Obtuse angle
3. Supplementary angles
4. Complementary angles
5. Adjacent angles

Research Statement

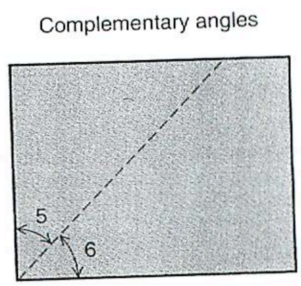
Findings from research studies suggest that students often have misconceived notions about angles and other geometric figures that are based solely on how these figures are oriented in textbooks.

Clements and Battista 1992

Solution **1, 2, 3, 5.** Any crease that intersects an edge of the paper forms supplementary angles with the edges. For example, the crease in figure (a) intersects $\overline{BC}$, forming supplementary angles 1 and 2. These angles are also adjacent angles. The same crease intersects $\overline{AB}$ and forms adjacent supplementary angles 3 and 4. Angles 1 and 4 are acute, and angles 2 and 3 are obtuse. **4, 5.** Any crease through a corner of the paper forms adjacent complementary angles with the edges. Angles 5 and 6 in figure (b) are adjacent complementary angles.



(a)



(b)

Two intersecting lines form four pairs of adjacent supplementary angles. For example, ∡1 and ∡4 in Figure 9.11 are supplementary angles. Nonadjacent angles formed by two intersecting lines, such as ∡2 and ∡4 in Figure 9.11, are called **vertical angles.**

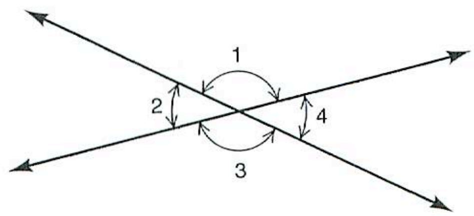


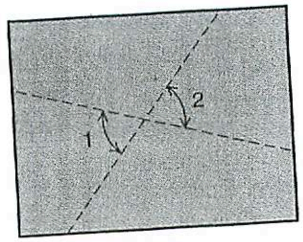
Figure 9.11

EXAMPLE D

1. Name four pairs of supplementary angles in Figure 9.11.
2. Name two pairs of vertical angles in Figure 9.11.
3. Fold a sheet of paper to create a model of two intersecting lines. Compare the measures of the vertical angles and make a conjecture.

Solution **1.** The following pairs of angles are supplementary angles: ∡1 and ∡4; ∡1 and ∡2; ∡2 and ∡3; ∡3 and ∡4. **2.** The following pairs of angles are vertical angles: ∡2 and ∡4; ∡1 and ∡3

3. Two intersecting creases produce vertical angles. Angles 1 and 2 in the figure here are vertical angles. Vertical angles are congruent. This can be illustrated by folding the angles onto each other.

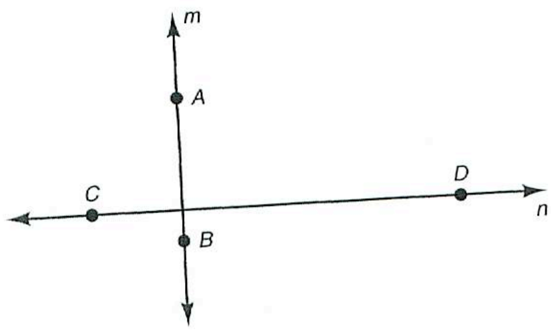


PERPENDICULAR AND PARALLEL LINES

NCTM Standards

. . . students in grades 3–5 should be expanding their mathematical vocabulary. . . . As they describe shapes, they should hear, understand, and use mathematical terms such as parallel, perpendicular, face, edge, vertex, angle, trapezoid, prism, and so forth, to communicate geometric ideas with greater precision. p. 166

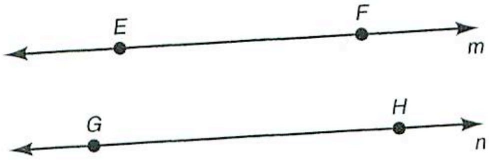
If two lines intersect to form right angles, they are **perpendicular.** Lines m and n in Figure 9.12 are perpendicular; this is indicated by writing $m \perp n$. Two line segments, such as $\overline{AB}$ and $\overline{CD}$ in Figure 9.12 are perpendicular if they lie on perpendicular lines. In this case, we write $\overline{AB} \perp \overline{CD}$ or $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$.



Perpendicular lines

Figure 9.12

If two lines are in a plane and they do not intersect, they are **parallel.** Lines m and n in Figure 9.13 are parallel; this is indicated by writing $m \parallel n$. Similarly, two segments are parallel if they lie in parallel lines. For example, segments $\overline{EF}$ and $\overline{GH}$ in Figure 9.13 are parallel, and we write $\overline{EF} \parallel \overline{GH}$.



Parallel lines

Figure 9.13

If two lines ℓ and m are intersected by a third line t (see Figure 9.14), we call line t a **transversal.** Two very special angles are created on the alternate sides of the transversal and interior to lines ℓ and m (angles 1 and 2 in Figure 9.14). These angles are called **alternate interior angles.** If the two lines ℓ and m are parallel (as in Figure 9.14), *the alternate interior angles have the same measure.* The converse of this statement is also true: *If the alternate interior angles have the same measure,* lines ℓ and m are parallel. These statements are combined in the following property.

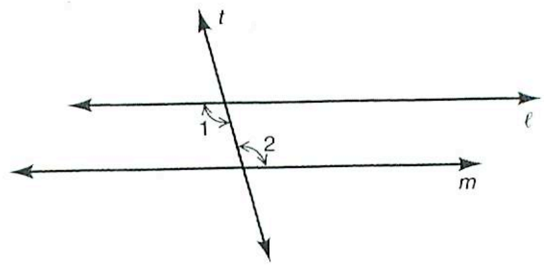


Figure 9.14

Alternate Interior Angles If two lines are intersected by a transversal, the lines are parallel if and only if the alternate interior angles created by the transversal have the same measure.

EXAMPLE E

Use a standard sheet of paper to model the following geometric terms: parallel lines, perpendicular lines, lines intersected by a transversal, and alternate interior angles having the same measure. Draw and label these on the paper.

Solution The opposite edges of the paper are parallel line segments, and any two edges that meet at a corner are perpendicular line segments. Any fold of the paper that intersects the opposite parallel edges of the paper will create alternate interior angles with the same measure.

There are other ways of obtaining parallel and perpendicular lines by folding paper. Two perpendicular lines can be obtained by folding the paper in half along one edge and then folding it in half along the other edge. Two parallel lines can be obtained by folding the paper in half along one edge and then folding it in half again along the same edge.

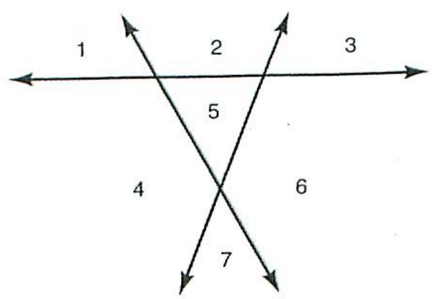
PROBLEM-SOLVING APPLICATION

Problem

What is the maximum number of regions into which a plane can be partitioned by 12 lines?

Understanding the Problem One line partitions a plane into 2 regions, and 2 intersecting lines partition a plane into 4 regions. **Question 1:** What is the maximum number of regions created by 3 lines in a plane?

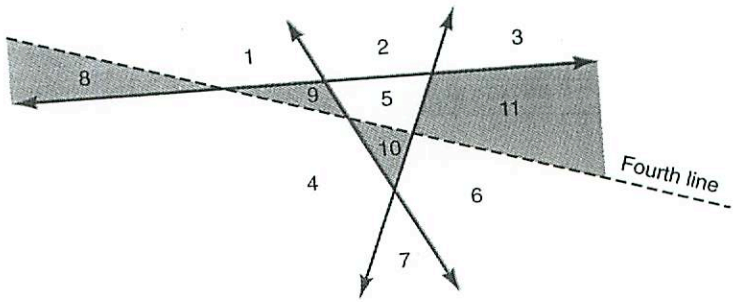
Devising a Plan It would be difficult to draw 12 lines and count the resulting regions. Let's *make a table* to record the numbers of regions for the first few lines. This approach may suggest a solution. Three lines divide the plane into 7 regions. **Question 2:** What is the maximum number of regions created by 4 lines?



Carrying Out the Plan The following table lists the maximum number of regions for 1, 2, 3, and 4 lines. Find a pattern and use inductive reasoning to predict the numbers of regions for the next few lines. **Question 3:** How many regions will there be for 12 lines?

No. of lines	1	2	3	4	5	6	7	8	9	10	11	12
No. of regions	2	4	7	11								

Looking Back When a fourth line that is not parallel to any of the first 3 lines is drawn on the plane, by definition it will intersect each of the 3 given lines. Also, it will cut across 4 regions, as shown in the figure below. This accounts for 4 new regions. **Question 4:** How many lines and how many regions will a fifth nonparallel line intersect?



Answers to Questions 1–4 **1.** 7 **2.** 11

3.

No. of lines	5	6	7	8	9	10	11	12
No. of regions	16	22	29	37	46	56	67	79

4. The fifth line will intersect 4 lines and 5 regions to create 5 new regions.

CURVES AND CONVEX SETS

We can draw a curve through a set of points by using a single continuous motion (Figure 9.15).

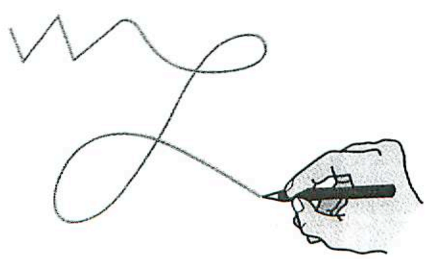


Figure 9.15

Several types of curves are shown in Figure 9.16. Curve A is called a **simple curve** because it starts and stops without intersecting itself. Curve B is a **simple closed curve** because it is a simple curve that starts and stops at the same point. Curve C is a **closed curve,** but since it intersects itself, it is not a simple closed curve.

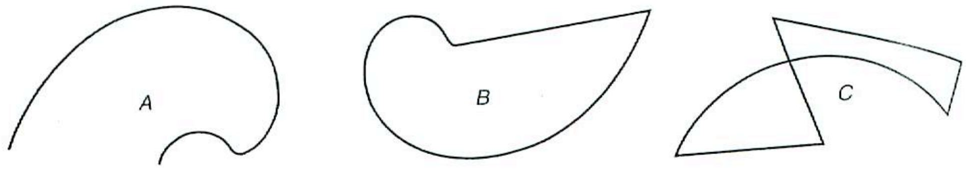
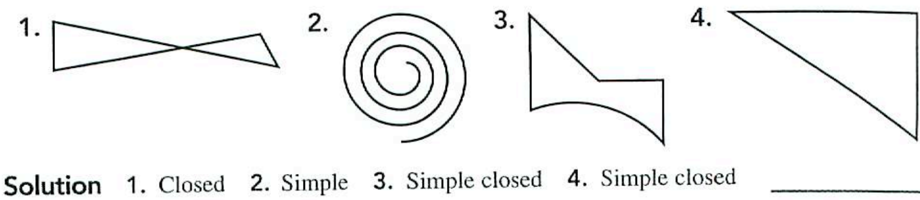


Figure 9.16

EXAMPLE F Classify each curve as simple, simple closed, or closed.



Solution **1.** Closed **2.** Simple **3.** Simple closed **4.** Simple closed

A well-known theorem in mathematics, called the **Jordan curve theorem,** states that every simple closed curve partitions a plane into three disjoint sets: the points on the curve, the points in the interior, and the points in the exterior. This means that if K is in the interior and M is in the exterior, then $\overline{KM}$ will intersect the curve (Figure 9.17).

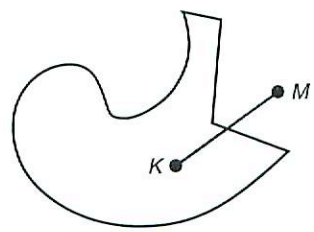
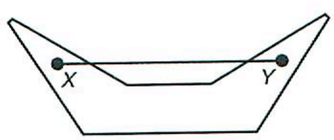
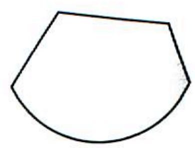


Figure 9.17

Convex Sets The union of a simple closed curve and its interior is called a **plane region.** Plane regions can be classified as *concave* or *convex*. You may have heard the word *concave*. It is from the Latin word *concavus,* meaning hollow. Intuitively, a concave set may be thought of as "caved in," as in Figure 9.18a. To be more mathematically precise, we say that a set is **concave** if it contains two points such that the line segment joining the points does not completely lie in the set. The set in Figure 9.18a is concave because $\overline{XY}$ is not completely in the set. If a set is not concave, it is called **convex.** An intuitive way of thinking about a convex set is to imagine enclosing the boundary of a figure with an elastic band. If the elastic touches all points on the boundary, as it will for the set in Figure 9.18b, the set is convex; and if not, as in Figure 9.18a, the set is concave (also sometimes called **nonconvex**).



Concave
(a)

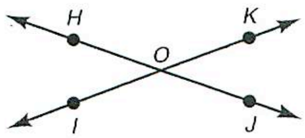


Convex
(b)

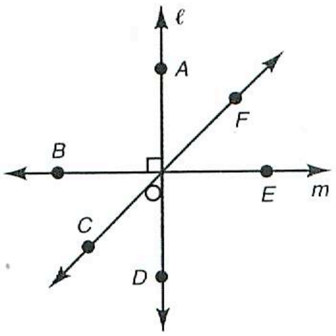
Figure 9.18

Use the angles in the figures in exercises 9 and 10 to identify the pairs of angles.

9. **a.** Three pairs of adjacent supplementary angles
b. Two pairs of vertical angles
c. Two pairs of angles with the same measure

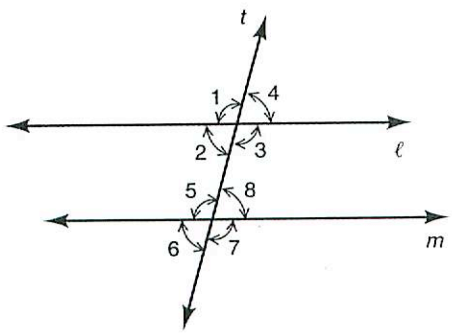


10. **a.** Three pairs of adjacent supplementary angles
b. Three pairs of vertical angles
c. Two pairs of adjacent complementary angles



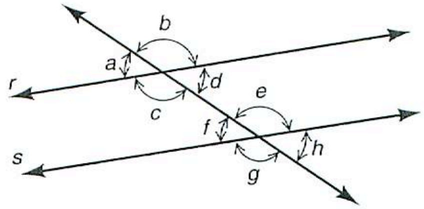
11. Draw a circle that illustrates each of the following geometric situations.
a. A diameter that is perpendicular to a chord
b. A line tangent to the circle at one end of a radius
c. Two chords that bisect each other

12. If ℓ and m are parallel lines, explain why the angles in each pair in parts a through d have the same measure.

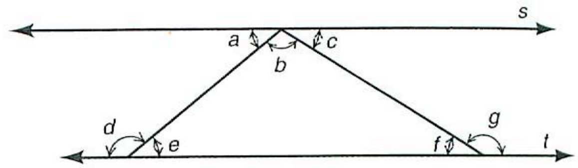


a. $\measuredangle 2$ and $\measuredangle 8$
b. $\measuredangle 2$ and $\measuredangle 4$
c. $\measuredangle 4$ and $\measuredangle 8$ (These angles are called **corresponding angles.**)
d. $\measuredangle 1$ and $\measuredangle 7$
e. Explain why $\measuredangle 3$ and $\measuredangle 8$ are supplementary angles.

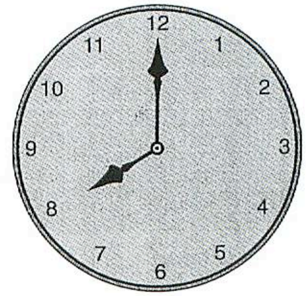
13. If r and s are parallel lines and the measure of $\measuredangle a$ is 34.5°, what is the measure of each of the following angles?
a. $\measuredangle e$ **b.** $\measuredangle h$
c. $\measuredangle c$ **d.** $\measuredangle f$



14. If s and t are parallel lines and the measure of $\measuredangle f$ is 32° and the measure of $\measuredangle a$ is 40°, determine the measure of each of the following angles.
a. $\measuredangle b$ **b.** $\measuredangle c$ **c.** $\measuredangle d$ **d.** $\measuredangle e$ **e.** $\measuredangle g$



Use the type of clock shown here to answer the questions in exercises 15 and 16.

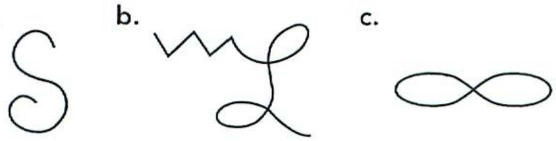


15. **a.** What is the measure of the obtuse angle formed by the hour and minute hands of a clock if the time is 8 o'clock?
b. How many degrees will the hour hand of the clock move through when the time changes from 8 o'clock to 10 o'clock?
c. How many minutes have passed when the minute hand has moved through 42°?

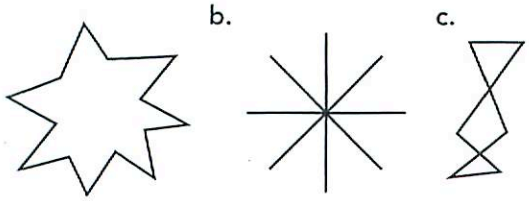
16. **a.** How many degrees will the minute hand of the clock move through when the time changes from 8 o'clock to 8:25?
b. How many hours will have passed when the hour hand has moved through 120°?
c. What is the measure of the obtuse angle formed by the hour hand and the minute hand if the time is 2:30?

Classify each curve in exercises 17 and 18 as simple, simple closed, closed, or none of these.

17. a. b. c.

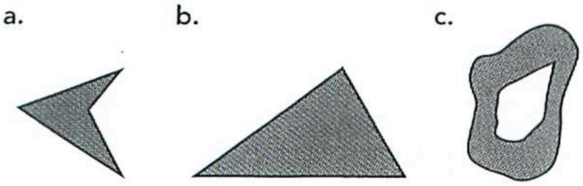


18. a. b. c.

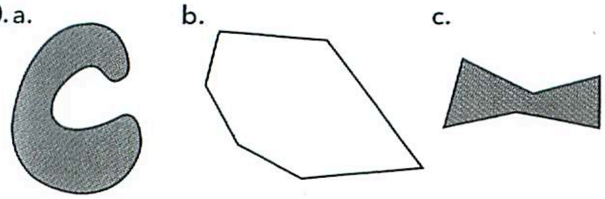


Classify each region in exercises 19 and 20 as convex or concave.

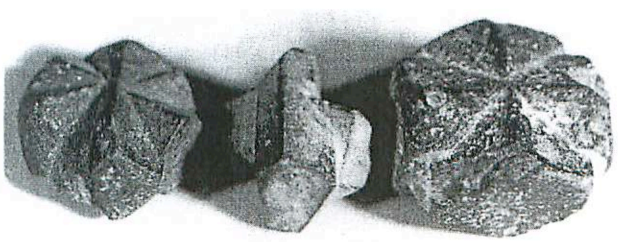
19. a. b. c.



20. a. b. c.



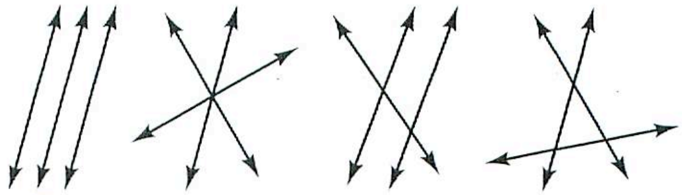
21. The following photograph shows three crystals of the mineral staurolite. These crystals are found in all parts of the world. They are especially common in the Shenandoah Valley. The crystal on the far right is known as the *Fairy Stone* of the Appalachian Mountains. This form and the one on the left are often imitated by jewelers.



a. The ridges on the top of the Fairy Stone form two lines that intersect to form four angles of equal measure. What is the measure of each of these angles?

b. The ridges on the top of the crystal on the left form three lines that intersect to form six angles of equal measure. What is the measure of each of these angles?

Three lines in a plane may intersect in 0, 1, 2, or 3 points. In exercises 22 and 23, draw lines to support your reasoning.



22. Determine all the different numbers of points of intersection that are possible with four lines in a plane.

23. Determine all the different numbers of points of intersection that are possible with five lines in a plane.

Draw some figures in exercises 24 and 25 to determine whether the following statements are true or false. For each false statement show a counterexample.

24. a. The two diagonals of a parallelogram have the same length.

b. Any two angles in a parallelogram that share a common side are supplementary.

c. The two diagonals of a rectangle have the same length.

25. a. If the two diagonals in a parallelogram have the same length, the parallelogram is a rectangle.

b. If the midpoints of the adjacent sides of a rectangle are connected, another rectangle is formed.

c. If the midpoints of the adjacent sides of a quadrilateral are connected, a parallelogram is formed.

Reasoning and Problem Solving

26. PS To prepare for their annual volleyball party, the Chase family has laid out a four-sided volleyball court in which two opposite sides have a length of 30 feet each and the other two opposite sides have a length of 60 feet each.

a. Explain why the court may not be rectangular. What shape might it have?

b. Which statement in exercises 24 or 25 can be used to determine if the court is rectangular?

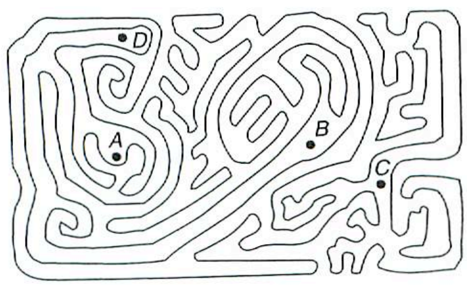
27. PS A 70-inch piece of pipe is to be cut at two points A and B such that A and B are not on the ends of the pipe and the length from A to B is 42 inches. How many possibilities are there for obtaining three pieces of pipe of different lengths if the lengths are whole numbers?

28. PS **Featured Strategies: Solving a Simpler Problem and Making a Table.** What is the maximum number of points of intersection for 12 lines?

a. **Understanding the Problem.** The problem asks for the greatest possible number of points of intersection. What is the minimum number of points of intersection for 12 lines?

c. Draw line segment $\overline{AB}$. Count the number of times that $\overline{AB}$ intersects the curve in (i). How can this number be used to tell when a point is inside or outside a simple closed curve? (*Hint:* Draw a few simple closed curves.) Check your answer by drawing $\overline{CD}$ for the curve in (ii).

33. The following simple closed curve is from *Puzzles and Graphs* by John Fujii.* Determine whether the points in each pair below are on the same side of the curve.



a. *A*, *B* b. *B*, *C* c. *D*, *C* d. *B*, *D*

34. An equilateral triangle has three sides of equal length. Fold a sheet of paper to form an equilateral triangle, using the creases or edges of the paper. (*Hint:* Obtain a centerline by folding the paper in half.)

Writing and Discussion

1. A student who built a pentagonal pattern block figure, as in question 3 in the one-page **Math Activity** in this section, says that the figure has seven sides. How do you think she got seven and how would you help her so she doesn't make that mistake again?

2. Students are sorting polygonal shapes into two groups, parallelograms and nonparallelograms. You see several students putting squares and rectangles into the nonparallelogram group. Describe ways you can help these students understand that squares and rectangles are parallelograms.

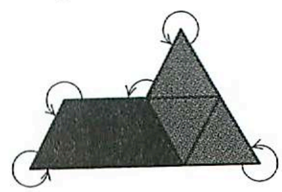
3. During a paper-folding activity in your class, a student concluded that any two line segments that do not intersect must be parallel. Is this student correct? How would you respond?

4. Pierre van Hiele and Dieke van Hiele-Geldof were teachers in the Netherlands who developed a theory about how students learn geometry. The "van Hiele theory"—that students learn geometry by progressing through five stages or levels of reasoning—prompted much research on the teaching and learning of geometry. Find information about the van Hiele levels of reasoning in geometry and describe each level and how the levels differ.

Making Connections

1. Read the first two recommendations for the **Grades 3–5 Standards—Geometry** (see inside front cover). Describe ways that you can help students in grades 3–5 begin to understand abstract relationships among quadrilaterals. As examples: all squares are rectangles but not all rectangles are squares; squares are all rhombuses, but not all rhombuses are squares; all squares, rhombuses, and rectangles are parallelograms, etc.

2. In the one-page **Math Activity** at the beginning of this section you discovered a way to predict the sum of the measures of the interior angles of a polygon. Use the pattern block figures on that page to predict the sum of the measures of the "outside angles" of any polygon. An example of the five outside angles of a pentagon is shown in this diagram.



3. Assume that you have a classroom set of circular protractors and have taught students to measure angles. The **Standards** statement on page 572 suggests that students learn to estimate angle measure. Design a two-person game that will help students learn to estimate angle measures. Write rules for the game in which students will check their estimates using the protractor.

4. Read the **Research** statement on page 574. Explain what you believe the statement means for angles and figures. Include sketches of figures with your explanation.

*John Fujii, *Puzzles and Graphs* (Reston, VA. National Council of Teachers of Mathematics, 1966).